

COHOMOLOGY OF WHEELS ON TORIC VARIETIES

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ABSTRACT. We describe explicitly the cohomology of the total complex of certain diagrams of invertible sheaves on normal toric varieties. These diagrams, called wheels, arise in the study of toric singularities associated to dimer models. Our main tool describes the generators in a family of syzygy modules associated to the wheel in terms of walks in a family of graphs.

1. INTRODUCTION

The purpose of this paper is to compute explicitly the cohomology of certain complexes of invertible sheaves on normal toric varieties. Each complex that we consider is the total complex

$$T^\bullet: \quad T^{-3} \xrightarrow{d^3} T^{-2} \xrightarrow{d^2} T^{-1} \xrightarrow{d^1} T^0 \quad (1.1)$$

of a diagram of the form

$$\quad (1.2)$$

for some $m \geq 2$, where L , $L_{j,j+1}$ and L_j ($1 \leq j \leq m$) are invertible sheaves on a normal toric variety X , and where the maps satisfy natural relations that makes T^\bullet into a complex (see Section 3). Here, the labelling of an arrow by a divisor D indicates that the map is $-\otimes_{\mathcal{O}_X}(D)$, and the underscore means that the right-hand copy of L lies in degree zero. Since the invertible sheaves at the extreme left and right of the diagram coincide, the diagram can be represented equally well in a planar picture that is reminiscent of a bicycle wheel (see Figure 5 in Section 3). As a result, we refer to every such diagram as a ‘wheel’ of invertible sheaves on X .

The main result of this paper calculates the cohomology of T^\bullet explicitly, generalising to arbitrary $m \geq 2$ the result for $m = 3$ by Cautis–Logvinenko [3, Lemma 3.1] (the proof from *loc.cit.* is valid as stated if the smooth scheme under consideration is a toric variety). To state

the result more precisely, we choose an order on the set of transpositions of m letters, giving $\tau_1 = (\mu_1, \nu_1), \dots, \tau_n = (\mu_n, \nu_n)$ where $n = \binom{m}{2}$ and $\mu_k < \nu_k$ for $1 \leq k \leq n$. In addition, for every index $1 \leq k \leq n$ we define a subscheme $Z_k \subset X$ to be the scheme-theoretic intersection of certain torus-invariant divisors in X (see Definition 3.2). Our main result can be stated as follows (see Theorem 3.3):

Theorem 1.1. *Let T^\bullet be the complex as above. Then:*

- (1) $H^0(T^\bullet) \cong \mathcal{O}_Z \otimes L$ where Z is the scheme-theoretic intersection of D^1, \dots, D^m ;
- (2) $H^{-1}(T^\bullet)$ has an n -step filtration $\text{im}(d^2) = F^0 \subseteq F^1 \subseteq \dots \subseteq F^{n-1} \subseteq F^n = \ker(d^1)$ where, for $1 \leq k \leq n$ and for the permutation $\tau_k = (\mu_k, \nu_k)$, we have

$$F^k / F^{k-1} \cong \mathcal{O}_{Z_k} \otimes L_{\mu_k} \otimes L_{\nu_k} \otimes L^{-1}(\text{gcd}(D^{\mu_k}, D^{\nu_k})); \quad (1.3)$$

- (3) $H^{-2}(T^\bullet) \cong \mathcal{O}_D \otimes L(D)$ where $D = \text{gcd}(D_{1,2}, D_{2,3}, \dots, D_{m,1})$;
- (4) $H^{-3}(T^\bullet) \cong 0$.

Our motivation to understand precisely the cohomology of the complex T^\bullet comes from the study of derived categories of certain smooth toric varieties associated to consistent dimer model algebras. Theorem 1.1 enables Bocklandt–Craw–Quintero-Vélez [2] to prove that the image under a derived equivalence of every indecomposable simple module (with one exception) is concentrated in one degree, thereby extending to the dimer setting a result of Cautis–Logvinenko [3] and Cautis–Craw–Logvinenko [4] for any finite abelian subgroup of $\text{SL}(3, \mathbb{C})$.

In order to prove Theorem 1.1, we realise the total complex T^\bullet as a complex of $\text{Cl}(X)$ -graded modules over the Cox ring S of X , namely

$$S(L^{-1}) \xrightarrow{\varphi^3} \bigoplus_{j=1}^m S(L_{j,j+1}^{-1}) \xrightarrow{\varphi^2} \bigoplus_{j=1}^m S(L_j^{-1}) \xrightarrow{\varphi^1} S(L^{-1}), \quad (1.4)$$

where $S(L^{-1})$ denotes the free S -module with generator in degree $L \in \text{Cl}(X)$. This reduces the problem to one of commutative algebra. The lion's share of the effort in proving Theorem 1.1 goes into proving part (2). For this, the image of φ^2 is generated by elements $\alpha_1, \dots, \alpha_m$, and our chosen order on the set of transpositions on m letters determines an order on the generators β_1, \dots, β_n of $\ker(\varphi^1)$ which in turn defines a filtration

$$\text{im}(\varphi^2) = F^0 \subseteq F^1 \subseteq F^2 \subseteq \dots \subseteq F^{n-1} \subseteq F^n = \ker(\varphi^1).$$

We give a presentation for each successive quotient F^k / F^{k-1} as a cyclic $\text{Cl}(X)$ -graded S -module of the form $(S/I_k)(L_{\mu_k}^{-1} \otimes L_{\nu_k}^{-1} \otimes L)$ for some monomial ideal I_k whose generators are given in terms of the divisors D^1, \dots, D^m labelling the right-hand arrows in the diagram (1.2) illustrating the wheel (see Proposition 3.1). While Macaulay2 [7] makes this computation straightforward in any given example, we provide a unified description for all $1 \leq k \leq n$. (Warning: Macaulay2 may choose an order on the generators β_1, \dots, β_n that differs from ours.)

Our main tool, which may be of independent interest, is a description of the syzygy module of $\ker(\varphi^1)$ in terms of walks in the complete graph Γ on m vertices. In fact, for each $1 \leq k \leq n$ we introduce a subgraph Γ_k of Γ that enables us to describe uniformly the module of syzygies $\text{syz}(F^k)$ in terms of certain walks in Γ_k . To state the result, recall that a circuit in Γ_k is a closed walk that does not pass through a given vertex twice. It is straightforward to associate

a syzygy to every such circuit (see Lemma 2.3). A circuit is said to be minimal if it admits no chords (see (2.4)). We prove the following result Theorem 2.5.

Theorem 1.2. *For $m \leq k \leq n$, the module $\text{syz}(F^k)$ is generated by the set of syzygies associated to the minimal circuits of Γ_k .*

The precise description of the syzygies from Theorem 1.2 allows us to read off directly a set of monomial generators for each ideal I_k , and this feeds into the proof of Theorem 1.1 above. Generating sets for toric ideals arising from graphs were studied by Hibi–Ohsugi [10], and some of the graph-theoretic tools that we use here were also employed there. Properties of \mathbb{k} -algebras arising from graphs have also been studied widely by Villarreal, see for example [11].

Acknowledgements. Thanks to Raf Bocklandt for generating Example 3.5, and to Sonja Petrovic for comments on an earlier version of this paper. Our results owes much to experiments made with Macaulay2 [7]. Both authors were supported by EPSRC grant EP/G004048.

2. SYZYGIES FROM WALKS IN A COMPLETE GRAPH

Let S be a polynomial ring over a field \mathbb{k} and let $f^1, \dots, f^m \in S$ be monomials for some $m \geq 2$. Consider the free S -module with basis $\mathbf{e}_1, \dots, \mathbf{e}_m$ and define an S -module homomorphism

$$\varphi: \bigoplus_{\mu=1}^m S\mathbf{e}_\mu \longrightarrow S$$

by setting $\varphi(\mathbf{e}_\mu) = f^\mu$ for $1 \leq \mu \leq m$. For every pair of indices $1 \leq \mu < \nu \leq m$ we define monomials $f^{\mu,\nu} = \text{lcm}(f^\mu, f^\nu)$ and set

$$\beta_{(\mu,\nu)} = \frac{f^{\mu,\nu}}{f^\nu} \mathbf{e}_\nu - \frac{f^{\mu,\nu}}{f^\mu} \mathbf{e}_\mu. \quad (2.1)$$

The module of syzygies of $M := \langle f^1, \dots, f^m \rangle$ is defined to be the S -module $\text{syz}(M) := \ker(\varphi)$. The following result is well known; see for example Eisenbud [6, Lemma 15.1].

Lemma 2.1. *The kernel of φ is generated by the elements $\beta_{(\mu,\nu)}$ for $1 \leq \mu < \nu \leq m$.*

It is convenient to order the set $\{(\mu, \nu) \mid 1 \leq \mu < \nu \leq m\}$ of transpositions of m letters. First list the transpositions of adjacent letters $\tau_j = (j, j+1)$ for $1 \leq j \leq m-1$. Set $\tau_m = (1, m)$, then list all remaining transpositions that involve 1 as $\tau_j = (1, j-m+2)$ for $m+1 \leq j \leq 2m-3$, and finally list all remaining transpositions lexicographically, so $\tau_i = (\mu_i, \nu_i)$ precedes $\tau_j = (\mu_j, \nu_j)$ if and only if $\mu_i < \mu_j$ or $\mu_i = \mu_j$ and $\nu_i < \nu_j$. We may therefore list the generators of $\ker(\varphi)$ from Lemma 2.1 by setting $\beta_j := \beta_{(\mu_j, \nu_j)}$ for all $1 \leq j \leq n$, where $n = \binom{m}{2}$. This choice of order enables us to define for each $1 \leq k \leq n$ an S -module

$$F^k = \langle \beta_1, \dots, \beta_k \rangle.$$

Our primary goal is to provide for each $1 \leq k \leq n$ an explicit set of generators for the module of syzygies $\text{syz}(F^k)$ that encodes the relations between β_1, \dots, β_k . Recall that this module is defined to be the kernel of the surjective S -module homomorphism

$$\psi: \bigoplus_{j=1}^k S\mathbf{e}_j \longrightarrow F^k$$

satisfying $\psi(\mathbf{e}_j) = \beta_j$ for $1 \leq j \leq k$. We compute this module directly for $1 \leq k \leq m$.

Lemma 2.2. *The S -module $\text{syz}(F^k)$ is the zero module for $1 \leq k \leq m-1$, and it is a free module of rank one for $k = m$.*

Proof. Our choice of order on transpositions ensures that for $1 \leq k \leq m-1$, there can be no relations between β_1, \dots, β_k . For $k = m$, let $\sigma = \sum_{j=1}^m s_j \varepsilon_j$ be a syzygy on β_1, \dots, β_m where $s_1, \dots, s_m \in S$. By comparing coefficients in

$$0 = \psi(\sigma) = s_m \left(\frac{f^{1,m}}{f^m} \mathbf{e}_m - \frac{f^{1,m}}{f^1} \mathbf{e}_1 \right) + \sum_{j=1}^{m-1} s_j \left(\frac{f^{j,j+1}}{f^{j+1}} \mathbf{e}_{j+1} - \frac{f^{j,j+1}}{f^j} \mathbf{e}_j \right)$$

we obtain the following equations

$$s_1 f^{1,2} = s_2 f^{2,3} = \dots = s_{m-1} f^{m-1,m} = -s_m f^{1,m}. \quad (2.2)$$

It's easy to see (or see Lemma 2.3 below for a proof) that the element

$$\sigma_0 := -\frac{\text{lcm}(f^1, \dots, f^m)}{f^{1,m}} \varepsilon_m + \sum_{j=1}^{m-1} \frac{\text{lcm}(f^1, \dots, f^m)}{f^{j,j+1}} \varepsilon_j \quad (2.3)$$

is a syzygy. Moreover, equations (2.2) imply that

$$\sigma = \frac{s_1 f^{1,2}}{\text{lcm}(f^1, \dots, f^m)} \sigma_0,$$

so $\text{syz}(F^m)$ is the free S -module with basis σ_0 . \square

We study the module $\text{syz}(F^k)$ for $m+1 \leq k \leq n$ by studying walks in a graph. Let Γ be the complete graph on m vertices, with vertex set $\{1, 2, \dots, m\}$. Assign an orientation to each edge $e = (\mu, \nu)$ by directing it from μ to ν if $\mu < \nu$. Regard every such edge as being labelled by the corresponding generator $\beta_{(\mu, \nu)}$ of $\ker(\varphi)$. The order on the generators β_1, \dots, β_n introduced above determines an order on the set of edges e_1, \dots, e_n of Γ . A *walk* γ of length ℓ in Γ is a walk in the undirected graph that traverses precisely ℓ edges. Every such walk is characterised by the sequence of vertices $\gamma = (\mu_1, \mu_2, \dots, \mu_{\ell+1})$ in Γ that it touches. A walk γ is *closed* if $\mu_1 = \mu_{\ell+1}$, and a *circuit* is a closed walk for which μ_1, \dots, μ_ℓ are distinct. Each circuit γ defines uniquely a subgraph of Γ , and we let $\text{supp}(\gamma)$ denote its set of edges. Given a circuit γ and an edge $e \in \text{supp}(\gamma)$, set $\text{sign}_\gamma(e) = +1$ if γ traverses e according to the orientation in Γ , and set $\text{sign}_\gamma(e) = -1$ if γ traverses e against orientation.

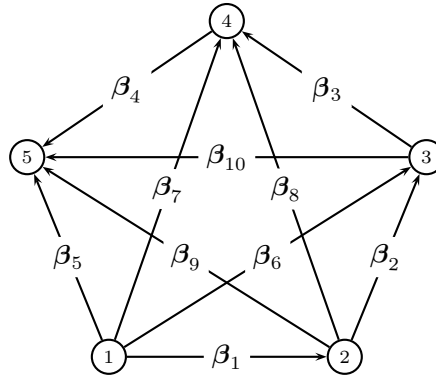


FIGURE 1. Directed graph Γ illustrating generators β_1, \dots, β_n for $m = 5$

Recall the map $\psi: \bigoplus_{j=1}^k S\epsilon_j \rightarrow F^k$. Given that the elements β_j for $1 \leq j \leq k$ correspond to edges in Γ , we may index the basis elements ϵ_j for $1 \leq j \leq k$ by edges e_1, \dots, e_k in Γ . Thus, for the edge $e = e_j$ for $1 \leq j \leq k$, we write $\epsilon_e := \epsilon_j$. For any vertices $\mu_1, \dots, \mu_{\ell+1}$ in Γ , set

$$f^{\mu_1, \dots, \mu_{\ell+1}} = \text{lcm}(f^{\mu_1}, \dots, f^{\mu_{\ell+1}}),$$

where $f^1, \dots, f^m \in S$ are the monomials encoded by the generators of the submodule M . For a walk $\gamma = (\mu_1, \mu_2, \dots, \mu_{\ell+1})$ in Γ we define the monomial $f^\gamma := f^{\mu_1, \dots, \mu_{\ell+1}}$. In particular, for an edge e in Γ joining vertex μ to ν , we obtain $f^e = f^{\mu, \nu}$.

Lemma 2.3. *For any circuit γ of length at least three in Γ , the vector*

$$\sigma_\gamma = \sum_{e \in \text{supp}(\gamma)} \text{sign}_\gamma(e) \frac{f^\gamma}{f^e} \epsilon_e$$

is a syzygy on β_1, \dots, β_n .

Proof. Note first that if γ has length two then $\sigma_\gamma = \epsilon_e - \epsilon_e = 0$ which is not in fact a syzygy by definition. For any circuit γ of length at least three we must show that

$$\psi(\sigma_\gamma) = \sum_{e \in \text{supp}(\gamma)} \text{sign}_\gamma(e) \frac{f^\gamma}{f^e} \beta_e = 0.$$

For an edge e that γ traverses in the direction from vertex μ to vertex μ' , we have that

$$\text{sign}_\gamma(e) \frac{f^\gamma}{f^e} \beta_e = \frac{f^\gamma}{f^e} \left(\frac{f^e}{f^{\mu'}} \mathbf{e}_{\mu'} - \frac{f^e}{f^\mu} \mathbf{e}_\mu \right) = \frac{f^\gamma}{f^{\mu'}} \mathbf{e}_{\mu'} - \frac{f^\gamma}{f^\mu} \mathbf{e}_\mu.$$

The sum of all such terms over $e \in \text{supp}(\gamma)$ collapses as a telescoping series since γ is closed. \square

For $1 \leq k \leq n$, let Γ_k denote the subgraph of Γ that has vertex set $\{1, \dots, m\}$, and which includes only the first k edges of Γ (see Figure 2(a) below for the case $k = m + 3$). Clearly $\Gamma = \Gamma_n$. Let $\gamma = (\mu_1, \dots, \mu_\ell, \mu_1)$ be a circuit in Γ_k for some k . A *chord* of γ in Γ_k is any edge of the form $c = (\mu_r, \mu_s)$ for some $1 \leq r < s \leq \ell$ that does not lie in $\text{supp}(\gamma)$. Every such chord c splits γ into two circuits:

$$\gamma_1 = (\mu_r, \dots, \mu_s, \mu_r) \quad \text{and} \quad \gamma_2 = (\mu_1, \dots, \mu_r, \mu_s, \dots, \mu_\ell, \mu_1). \quad (2.4)$$

A circuit must have length at least four if it is to admit a chord. We define a *minimal circuit* of Γ_k to be a circuit of length at least three that has no chords.

Lemma 2.4. *Let γ be a circuit in Γ_k admitting a chord in Γ_k that splits γ into circuits γ_1 and γ_2 as in (2.4). Then the syzygy σ_γ is contained in the module generated by σ_{γ_1} and σ_{γ_2} .*

Proof. Let c be the chord. For $i = 1, 2$, let $\gamma_i \setminus c$ denote the walk obtained from γ_i by removing the edge c . Since $\text{sign}_{\gamma_1}(c) = -\text{sign}_{\gamma_2}(c)$ we may rewrite

$$\begin{aligned} \sigma_\gamma &= \text{sign}_{\gamma_1}(c) \frac{f^\gamma}{f^c} \epsilon_c + \text{sign}_{\gamma_2}(c) \frac{f^\gamma}{f^c} \epsilon_c + \sum_{e \in \text{supp}(\gamma)} \text{sign}_\gamma(e) \frac{f^\gamma}{f^e} \epsilon_e \\ &= \text{sign}_{\gamma_1}(c) \frac{f^\gamma}{f^c} \epsilon_c + \sum_{e \in \text{supp}(\gamma_1 \setminus c)} \text{sign}_{\gamma_1}(e) \frac{f^\gamma}{f^e} \epsilon_e + \text{sign}_{\gamma_2}(c) \frac{f^\gamma}{f^c} \epsilon_c + \sum_{e \in \text{supp}(\gamma_2 \setminus c)} \text{sign}_{\gamma_2}(e) \frac{f^\gamma}{f^e} \epsilon_e \\ &= \frac{f^\gamma}{f^{\gamma_1}} \sigma_{\gamma_1} + \frac{f^\gamma}{f^{\gamma_2}} \sigma_{\gamma_2}. \end{aligned}$$

It remains to note that $f^{\gamma_1} = f^{\mu_r, \dots, \mu_s}$ divides $f^\gamma = f^{\mu_1, \dots, \mu_\ell}$, and similarly, f^{γ_2} divides f^γ . \square

We are now in a position to establish the main result of this section.

Theorem 2.5. *For $1 \leq k \leq n$, the S -module $\text{syz}(F^k)$ is generated by the syzygies σ_γ , where γ is a minimal circuit of Γ_k .*

Proof. We distinguish three cases. The first case, in which $1 \leq k \leq m-1$, is straightforward: the graph Γ_k admits no circuits and $\text{syz}(F^k) = 0$ by Lemma 2.2, so the result holds.

We prove the second case, in which $m \leq k \leq 2m-3$, by induction. For $k = m$, Lemma 2.2 shows that the S -module $\text{syz}(F^m)$ is free with basis σ_0 from (2.3). The syzygy σ_{γ_0} associated to the unique minimal circuit $\gamma_0 = (1, 2, \dots, m, 1)$ in Γ_m coincides with σ_0 , so the statement holds for $k = m$. Assume the statement for Γ_{k-1} for any $m+1 \leq k \leq 2m-3$, and let

$$\sigma = \sum_{j=1}^k s_j \varepsilon_j$$

be a syzygy on β_1, \dots, β_k where $s_1, \dots, s_k \in S$. As a first step we reduce to the case in which the coefficients satisfy $s_j = 0$ for $k-m+2 \leq j \leq m$ (these indices determine the edges to the left of β_k in Figure 2(a)). Indeed, suppose otherwise, so $s_i \neq 0$ for some $k-m+2 \leq i \leq m$.

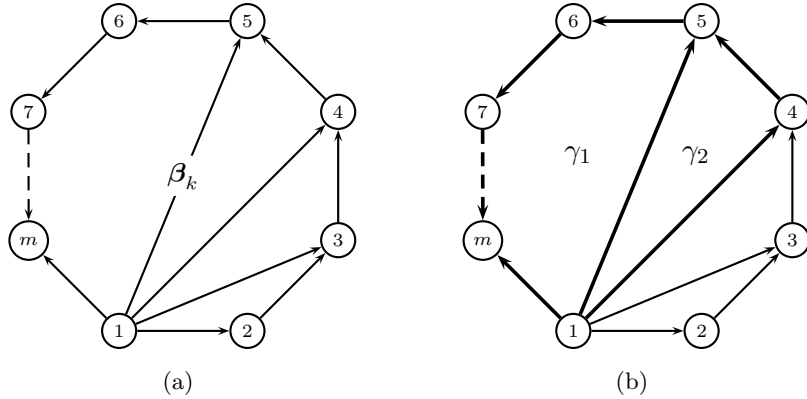


FIGURE 2. The graph Γ_k for $m \leq k \leq 2m-3$ illustrated for $k = m+3$

By comparing the coefficient of \mathbf{e}_μ for each index $k-m+3 \leq \mu \leq m$ in the equation

$$0 = \psi(\sigma) = \sum_{j=1}^k s_j \left(\frac{f^{\mu_j, \nu_j}}{f^{\nu_j}} \mathbf{e}_{\nu_j} - \frac{f^{\mu_j, \nu_j}}{f^{\mu_j}} \mathbf{e}_{\mu_j} \right),$$

we obtain a collection of equations

$$s_{k-m+2} f^{k-m+2, k-m+3} = s_{k-m+3} f^{k-m+3, k-m+4} = \dots = s_{m-1} f^{m-1, m} = -s_m f^{1, m} \quad (2.5)$$

which imply that $s_j \neq 0$ for all $k-m+2 \leq j \leq m$. As illustrated in Figure 2(b) for $k = m+3$, the circuit $\gamma_1 := (1, k-m+2, k-m+3, \dots, m, 1)$ is minimal in Γ_k , and it determines both the monomial $f^{\gamma_1} = f^{1, k-m+2, k-m+3, \dots, m}$ and the syzygy

$$\sigma_{\gamma_1} = -\frac{f^{\gamma_1}}{f^{1, m}} \varepsilon_m + \frac{f^{\gamma_1}}{f^{1, k-m+2}} \varepsilon_k + \sum_{j=k-m+2}^{m-1} \frac{f^{\gamma_1}}{f^{j, j+1}} \varepsilon_j. \quad (2.6)$$

Equations (2.5) and the fact that $s_m \neq 0$ imply that f^{γ_1} divides $s_m f^{1,m}$, and a straightforward computation shows that

$$\sigma_1 := \sigma - \frac{s_m f^{1,m}}{f^{\gamma_1}} \sigma_{\gamma_1} = \left(s_k - \frac{s_m f^{1,m}}{f^{1,k-m+2}} \right) \epsilon_k + \sum_{j=1}^{k-m+1} s_j \epsilon_j + \sum_{j=m+1}^{k-1} s_j \epsilon_j.$$

In particular, if we expand $\sigma_1 = \sum_{j=1}^k t_j \epsilon_j$ for $t_1, \dots, t_k \in S$, then $t_j = 0$ for $k-m+2 \leq j \leq m$, and it suffices to prove the result for σ_1 as claimed. The second step is to repeat the above, comparing the coefficient of ϵ_{k-m+2} in the equation $\psi(\sigma_1) = 0$, and since $t_{k-m+2} = 0$ we obtain

$$t_{k-m+1} f^{k-m+1, k-m+2} + t_k f^{1, k-m+2} = 0. \quad (2.7)$$

If $t_k \neq 0$ then the minimal circuit $\gamma_2 := (1, k-m+2, k-m+1, 1)$ in Γ_k from Figure 2(b) determines both the monomial $f^{\gamma_2} = f^{1, k-m+1, k-m+2}$ and the syzygy

$$\sigma_{\gamma_2} = \frac{f^{\gamma_2}}{f^{1, k-m+2}} \epsilon_k - \frac{f^{\gamma_2}}{f^{k-m+1, k-m+2}} \epsilon_{k-m+1} - \frac{f^{\gamma_2}}{f^{1, k-m+1}} \epsilon_{k-1}. \quad (2.8)$$

Equation (2.7) implies that f^{γ_2} divides $t_k f^{1, k-m+2}$ and again, a straightforward computation, this time using equation (2.7), shows that the coefficients of both ϵ_k and ϵ_{k-m+1} in the syzygy

$$\sigma_2 := \sigma_1 - \frac{t_k f^{1, k-m+2}}{f^{\gamma_2}} \sigma_{\gamma_2}$$

are zero. This means that $\sigma_2 \in \text{syz}(F^{k-1})$, and we deduce from the inductive hypothesis that σ_2 is generated by the elements σ_γ associated to minimal circuits γ in Γ_{k-1} . Among all minimal circuits in Γ_{k-1} , only $\gamma = (1, k-m+1, k-m+2, \dots, m, 1)$ is not minimal in Γ_k ; indeed, the edge labelled β_k is a chord. However, this edge splits γ into the circuits γ_1, γ_2 defined earlier in the current proof that are minimal in Γ_k , and Lemma 2.4 writes σ_γ as an S -linear combination of σ_{γ_1} and σ_{γ_2} . Thus, the syzygy σ_2 , and hence both σ_1 and σ , are generated by the elements σ_γ associated to minimal circuits γ in Γ_k . This completes the proof for $m \leq k \leq 2m-3$.

Finally, consider $2m-2 \leq k \leq n$. Let $>$ denote the induced monomial order on the free S -module $\bigoplus_{\mu=1}^m S \mathbf{e}_\mu$ defined for $g, g' \in S$ and $1 \leq \mu, \nu \leq m$ by taking $g' \mathbf{e}_\nu > g \mathbf{e}_\mu$ if and only if $g' f^\nu > g f^\mu$ with respect to any monomial order on S , or $g' f^\nu = g f^\mu$ and $\nu > \mu$. It follows that for $1 \leq j \leq k$, the leading term of β_j with respect to this order is $f^{\mu_j, \nu_j} / f^{\nu_j} \mathbf{e}_{\nu_j}$. This implies that the S -vectors of critical pairs are the elements

$$S(\beta_i, \beta_j) = \frac{f^{\mu_i, \mu_j, \nu_j}}{f^{\mu_j, \nu_j}} \beta_j - \frac{f^{\mu_i, \mu_j, \nu_j}}{f^{\mu_i, \nu_j}} \beta_i$$

arising from all elements in $\mathbb{B}_k := \{(i, j) \mid 1 \leq i < j \leq k, \nu_i = \nu_j\}$ (see Kreuzer–Robbiano [8, Definition 2.5.1]). Substituting (2.1) into every S -vector ensures that the leading terms cancel by definition. Since any critical pair (i, j) corresponds to a pair of directed edges (μ_i, ν_j) and (μ_j, ν_j) in Γ_k , the S -vector can then be written as a multiple of the generator $\beta_{(\mu_i, \mu_j)}$ corresponding to the third directed edge from Figure 3. Indeed, if we choose the index $1 \leq h \leq k$ so that $\beta_h = \beta_{(\mu_i, \mu_j)}$, then we compute explicitly that the ‘standard expressions’ are

$$S(\beta_i, \beta_j) = - \frac{f^{\mu_i, \mu_j, \nu_i}}{f^{\mu_i, \mu_j}} \beta_h.$$

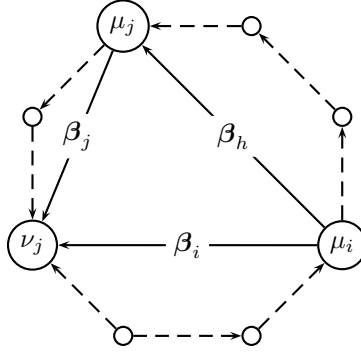


FIGURE 3. Minimal circuit in Γ_k for $2m - 2 \leq k \leq n$ where $i < j$.

Moreover, we deduce from Buchberger's Criterion [6, Theorem 15.8] that β_1, \dots, β_k are a Gröbner basis of F^k . Every standard expression determines a syzygy, namely

$$\sigma_{(i,j)} = \frac{f^{\mu_i, \mu_j, \nu_j}}{f^{\mu_j, \nu_j}} \varepsilon_j - \frac{f^{\mu_i, \mu_j, \nu_j}}{f^{\mu_i, \nu_j}} \varepsilon_i + \frac{f^{\mu_i, \mu_j, \nu_i}}{f^{\mu_i, \mu_j}} \varepsilon_h. \quad (2.9)$$

Schreyer's theorem [6, Theorem 15.10] implies that the set of syzygies $\{\sigma_{(i,j)} \mid (i,j) \in \mathbb{B}_k\}$ is a system of generators for $\text{syz}(F^k)$. Let $\gamma(i,j) := (\mu_i, \mu_j, \nu_j, \mu_i)$ denote circuit in Γ_k obtained by traversing the edges labelled β_h, β_j according to orientation followed by the edge labelled β_i against orientation (see Figure 3). Then $\sigma_{(i,j)}$ coincides with the syzygy $\sigma_{\gamma(i,j)}$ from Lemma 2.3, and the result is immediate from Lemma 2.6 to follow. \square

Lemma 2.6. *For $2m - 3 \leq k \leq n$, the minimal circuits in the graph Γ_k are precisely those of the form $\gamma(i,j) = (\mu_i, \mu_j, \nu_j, \mu_i)$ arising from pairs (i,j) in $\mathbb{B}_k = \{(i,j) \mid 1 \leq i < j \leq k, \nu_i = \nu_j\}$.*

Proof. We proceed by induction. Let γ be a minimal circuit in Γ_{2m-3} that is not of the form $\gamma(i,j)$ for any $(i,j) \in \mathbb{B}_{2m-3}$. Since γ is a circuit, it must traverse an edge e of the subgraph Γ_m , and since $\gamma \neq \gamma(i,j)$, then either the edge that follows e in γ , or that preceding e in γ , must lie in Γ_m . In either case, γ traverses two edges from Γ_m that share a common vertex μ . The special nature of Γ_{2m-3} then forces the edge $(1, \mu)$ to be a chord of γ , a contradiction. Assume now that the result holds for Γ_{k-1} and let γ be a minimal circuit in Γ_k that is not of the form $\gamma(i,j)$ for any $(i,j) \in \mathbb{B}_k$. If the edge $e_k = (\mu_k, \nu_k)$ does not lie in $\text{supp}(\gamma)$ then the result holds by induction, so we suppose otherwise. Let e be the unique edge in $\text{supp}(\gamma) \setminus \{e_k\}$ that has ν_k as a vertex. There are three cases:

- (i) $e = (\nu_k - 1, \nu_k)$, in which case (μ_k, ν_{k-1}) is a chord because $\gamma \neq \gamma(\nu_k - 1, k)$;
- (ii) $e = (\nu_k, \nu_k + 1)$, in which case γ must pass through a vertex of the form $1 \leq \mu \leq \mu_k$ since it is a circuit, but then (μ, ν_k) is a chord;
- (iii) $e = (\mu, \nu_k)$ for some $1 \leq \mu < \mu_k$. Since $\gamma \neq \gamma(j, k)$ for any $j < k$, the circuit γ must pass through another vertex of the form $1 \leq \mu' < \mu_k$, but then (μ', ν_k) is a chord.

Thus, the minimal circuit γ cannot exist. This completes the proof. \square

Remark 2.7. We now remark on the relation between Theorem 2.5 and existing results.

- (1) If for $2m - 2 \leq k \leq n$ we draw the vertices of Γ_k spaced evenly around a circle centred at the origin in \mathbb{R}^2 , then each minimal circuit γ has length three and hence determines a triangle as in Figure 3. In the spirit of the Taylor resolution of a monomial ideal

(see, for example, Bayer–Peeva–Sturmfels [1]), the triangle can be viewed as a 2-cell that defines f^{μ_i, μ_j, ν_j} , and the edges are 1-cells defining f^{μ_i, μ_j} , f^{μ_i, ν_j} and f^{μ_j, ν_j} . The coefficients of the syzygy $\sigma_{(i,j)}$ are then simply the quotients of the monomial for the 2-cell divided by the monomial for the corresponding 1-cell. An analogous statement holds for $m \leq k \leq 2m - 3$, where the syzygies σ_0 and σ_1 from the proofs of Lemma 2.2 and Theorem 2.5 respectively define polygons with more than three sides.

- (2) We emphasise that our choice of order on the set of transpositions of m letters is imposed on us by the geometry: the filtration in Proposition 3.1 below requires that the S -module F^k contains F^0 for $1 \leq k \leq n$. Without this constraint one could choose an alternative order in which each minimal circuit of Γ_k for $m \leq k \leq n$ determines a triangle, leading to a more unified proof of Theorem 2.5. Indeed, since f^1, \dots, f^m are monomials, the modules $\text{syz}(F^k)$ can be read off directly from the Taylor resolution for $1 \leq k \leq m$.

As an application of Theorem 2.5, we introduce a filtration of the module S -module $\ker(\varphi) = \text{syz}(M)$ that feeds into the proof of our main result. For $1 \leq k \leq n$, the S -modules F^k define a filtration

$$0 \subseteq F^1 \subseteq F^2 \subseteq \dots \subseteq F^{n-1} \subseteq F^n = \text{syz}(M)$$

in which the successive quotients are cyclic S -modules

$$\frac{F^k}{F^{k-1}} \cong \frac{\langle \beta_k \rangle}{\langle \beta_1, \dots, \beta_{k-1} \rangle \cap \langle \beta_k \rangle}. \quad (2.10)$$

The next result gives an explicit description of these quotient rings.

Proposition 2.8. *For each $1 \leq k \leq n$, the quotient F^k/F^{k-1} is isomorphic to the cyclic S -module S/I_k , where the monomial ideal I_k depends on k as follows:*

- (i) *for $1 \leq k \leq m - 1$, the ideal I_k is the zero ideal;*
- (ii) *for $k = m$, the ideal I_k is principal with generator $f^{1, \dots, m}/f^{1, m}$;*
- (iii) *for $m + 1 \leq k \leq 2m - 3$, the ideal is*

$$I_k = \left\langle \frac{f^{1, k-m+2, k-m+3, \dots, m}}{f^{1, k-m+2}}, \frac{f^{1, k-m+1, k-m+2}}{f^{1, k-m+2}} \right\rangle;$$

- (iv) *for $2m - 2 \leq k \leq n$, the corresponding transposition is $\tau_k = (\mu_k, \nu_k)$, and the ideal is*

$$I_k = \left\langle \frac{f^{\mu, \mu_k, \nu_k}}{f^{\mu_k, \nu_k}} \mid \mu \in \{1, \dots, \mu_k - 1\} \cup \{\nu_k - 1\} \right\rangle.$$

Proof. For $1 \leq k \leq n$, let $\{\sigma_1, \dots, \sigma_r\}$ be a set of generators for the S -module $\text{syz}(F^k)$. If we write $\sigma_\nu = \sum_{j=1}^k s_{\nu j} \epsilon_j$ with $s_{\nu 1}, \dots, s_{\nu k} \in S$ for $1 \leq \nu \leq r$, then [8, Proposition 3.2.3] implies that the coefficients s_{1k}, \dots, s_{rk} of ϵ_k generate the S -module $\langle \beta_1, \dots, \beta_{k-1} \rangle \cap \langle \beta_k \rangle$, so we obtain

$$\frac{F^k}{F^{k-1}} \cong \frac{S}{\langle s_{1k}, \dots, s_{rk} \rangle}.$$

It remains to compute $I_k := \langle s_{1k}, \dots, s_{rk} \rangle$. Parts (i) and (ii) now follow from Lemma 2.2 and equation (2.3). For part (iii), the proof of Theorem 2.5 shows that the only minimal circuits γ in Γ_k with $m + 1 \leq k \leq 2m - 3$ for which the associated syzygy σ_γ has a nonzero coefficient

for ε_k are $\gamma_1 := (1, k - m + 2, k - m + 3, \dots, m, 1)$ and $\gamma_2 := (1, k - m + 2, k - m + 1, 1)$. These nonzero coefficients are presented in equations (2.6) and (2.8), namely

$$\frac{f^{\gamma_1}}{f^{1, k-m+2}} = \frac{f^{1, k-m+2, k-m+3, \dots, m}}{f^{1, k-m+2}} \quad \text{and} \quad \frac{f^{\gamma_2}}{f^{1, k-m+2}} = \frac{f^{1, k-m+1, k-m+2}}{f^{1, k-m+2}}.$$

For part (iv), we deduce from Theorem 2.5 and Lemma 2.6 that $\text{syz}(F^k)$ is generated by the syzygies $\sigma_{(i,j)} = \sigma_{\gamma(i,j)}$ associated to pairs $(i, j) \in \mathbb{B}_k$. By equation (2.9), such syzygies have a nonzero coefficient of ε_k if and only if $(i, j) = (i, k)$ for those $1 \leq i < k$ satisfying $\nu_i = \nu_k$. The i th edge (μ_i, ν_i) in Γ_k has $\nu_i = \nu_k$ if and only if $\mu_i \in \{1, \dots, \mu_k - 1\} \cup \{\nu_k - 1\}$, that is, we must consider all pairs of the form (μ, ν_k) for $\mu \in \{1, \dots, \mu_k - 1\} \cup \{\nu_k - 1\}$. Equation (2.9) shows that the coefficient of ε_k in this case is $f^{\mu, \mu_k, \nu_k} / f^{\mu_k, \nu_k}$ as required. \square

Remark 2.9. The generators of I_k listed in Proposition 2.8 need not be minimal for $m + 1 \leq k \leq n$. For example (though not the simplest), a straightforward calculation for the module M over $S = \mathbb{k}[x_1, \dots, x_7]$ with generators

$$f^1 = x_1 x_6, \quad f^2 = x_1 x_2 x_7, \quad f^3 = x_2 x_3, \quad f^4 = x_3 x_4, \quad f^5 = x_4 x_5 x_7, \quad f^6 = x_5 x_6$$

gives $I_k = S$ for $k = 9, 10, 12, 13$. Thus, I_k is principal even though this ideal is listed as having more than one generator in Proposition 2.8.

3. COHOMOLOGY OF WHEELS ON TORIC VARIETIES

Let X be a normal toric variety over \mathbb{k} with Cox ring $S = \mathbb{k}[x_1, \dots, x_d]$. For an integer $m \geq 2$, consider the following diagram of invertible sheaves on X and maps between them:

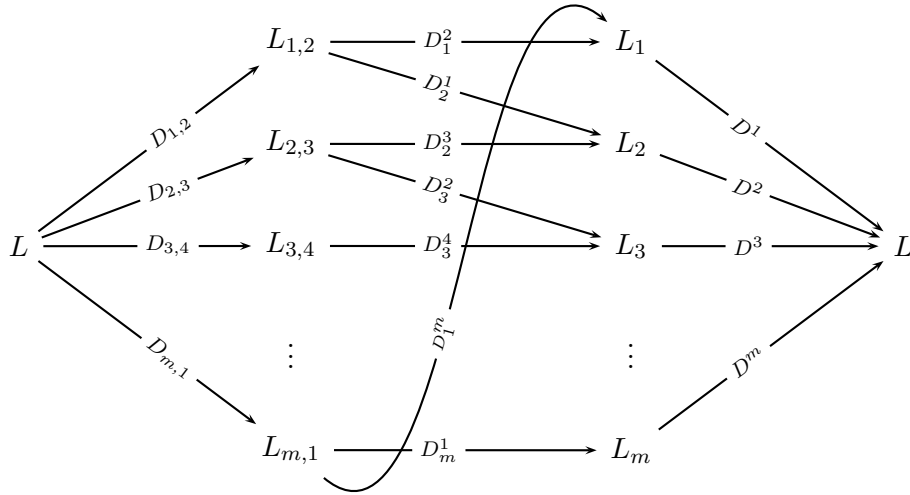


FIGURE 4. Invertible sheaves on X and maps between them

Here, the labelling of an arrow by a divisor D indicates that the map is $-\otimes \mathcal{O}_X(D)$. Thus, for example, we have that $\mathcal{O}_X(D_j^i) \cong L_{i,j}^{-1} \otimes L_j$. One can think of such diagram as a representation of a quiver (arising as the skeleton of a three-dimensional rhombic polyhedron) in the category of invertible sheaves on X . Throughout, we impose the condition that each of the two-dimensional rhombic faces of this quiver forms a commutative square, i.e.

$$D_{j+1}^j + D^{j+1} = D_j^{j+1} + D^j, \quad (3.1)$$

$$D_j^{j-1} + D_{j-1,j} = D_j^{j+1} + D_{j,j+1}, \quad (3.2)$$

for $1 \leq j \leq m$ (working modulo m from now on, where our indices lie in the range $1, \dots, m$). The relations (3.1) and (3.2) ensure that we can form the total chain complex T^\bullet , where

$$T^{-3} = L, \quad T^{-2} = \bigoplus_{j=1}^m L_{j,j+1}, \quad T^{-1} = \bigoplus_{j=1}^m L_j, \quad T^0 = L,$$

and $T^{-i} = 0$ for $i \neq -3, -2, -1, 0$. At each stage, the differential d is given by summing the maps in the diagram. Notice that the invertible sheaves at the far left-hand and right-hand of Figure 4 coincide. Thus, the invertible sheaves and the maps between them can be represented equally well in a planar picture; we call this the *wheel* of invertible sheaves on X , see Figure 5.

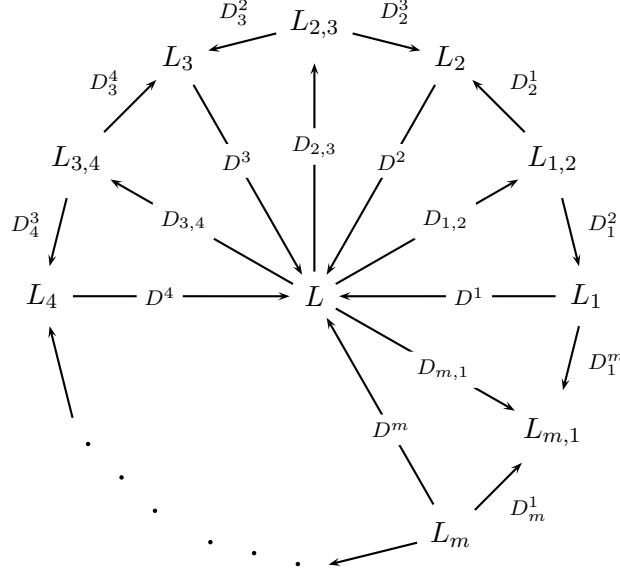


FIGURE 5. Wheel of invertible sheaves on X

We now use the results of the previous section to compute the cohomology of this complex. For this purpose, it is convenient to regard T^\bullet as a complex in the category of $\text{Cl}(X)$ -graded S -modules as follows. Let $S(L^{-1})$ denote the free S -module with generator \mathbf{e}_L in degree L , and for $1 \leq j \leq m$ let $S(L_j^{-1})$ and $S(L_{j,j+1}^{-1})$ denote the free S -modules with generators \mathbf{e}_j in degree L_j and $\mathbf{e}_{j,j+1}$ in degree $L_{j,j+1}$ respectively. In addition, we let f^j , f_{j+1}^j , f_j^{j+1} , and $f_{j,j+1}$ denote the monomials in S whose divisors of zeroes are the torus-invariant divisors D^j , D_{j+1}^j , D_j^{j+1} , and $D_{j,j+1}$. With this notation, T^\bullet is equivalent to the complex of $\text{Cl}(X)$ -graded S -modules

$$S(L^{-1}) \xrightarrow{\varphi^3} \bigoplus_{j=1}^m S(L_{j,j+1}^{-1}) \xrightarrow{\varphi^2} \bigoplus_{j=1}^m S(L_j^{-1}) \xrightarrow{\varphi^1} S(L^{-1}), \quad (3.3)$$

with differentials

$$\varphi^3(\mathbf{e}_L) = \sum_{j=1}^m f_{j,j+1} \mathbf{e}_{j,j+1}, \quad \varphi^2(\mathbf{e}_{j,j+1}) = f_{j+1}^j \mathbf{e}_{j+1} - f_j^{j+1} \mathbf{e}_j, \quad \varphi^1(\mathbf{e}_j) = f^j \mathbf{e}_L.$$

Notice that the map φ^1 is of the form considered in the preceding section, so we may list the generators of its kernel in a sequence β_1, \dots, β_n with $n = \binom{m}{2}$. We also list the generators of the image of φ^2 as

$$\alpha_j := f_{j+1}^j \mathbf{e}_{j+1} - f_j^{j+1} \mathbf{e}_j$$

for $1 \leq j \leq m$. The next proposition is central to the main result of this paper.

Proposition 3.1. *The S -modules*

$$F^k = \begin{cases} \langle \beta_1, \dots, \beta_k, \alpha_{k+1}, \dots, \alpha_m \rangle & \text{for } 1 \leq k \leq m, \\ \langle \beta_1, \dots, \beta_m, \beta_{m+1}, \dots, \beta_j \rangle & \text{for } m+1 \leq j \leq n, \end{cases}$$

define a filtration

$$\text{im}(\varphi^2) = F^0 \subseteq F^1 \subseteq \dots \subseteq F^{n-1} \subseteq F^n = \ker(\varphi^1).$$

Moreover, for each $1 \leq k \leq n$ and for the corresponding transposition is $\tau_k = (\mu_k, \nu_k)$, the quotient F^k/F^{k-1} is isomorphic to the cyclic $\text{Cl}(X)$ -graded S -module $(S/I_k)(L_{\mu_k}^{-1} \otimes L_{\nu_k}^{-1} \otimes L)$, where the monomial ideal I_k depends on k as follows:

(1) for $1 \leq k \leq m$, the ideal is

$$I_k = \left\langle \gcd(f_{k+1}^k, f_k^{k+1}), \frac{\text{lcm}(f^{1,\dots,m}, \gcd(f_{k+2}^{k+1}, f_{k+1}^{k+2}), \dots, \gcd(f_1^m, f_m^1))}{f^{k,k+1}} \right\rangle;$$

(2) for $m+1 \leq k \leq 2m-3$, the ideal is

$$I_k = \left\langle \frac{f^{1,k-m+2,k-m+3,\dots,m}}{f^{1,k-m+2}}, \frac{f^{1,k-m+1,k-m+2}}{f^{1,k-m+2}} \right\rangle;$$

(3) for $2m-2 \leq k \leq n$, the ideal is

$$I_k = \left\langle \frac{f^{\mu,\mu_k,\nu_k}}{f^{\mu_k,\nu_k}} \mid \mu \in \{1, \dots, \mu_k - 1\} \cup \{\nu_k - 1\} \right\rangle.$$

Proof. To prove that the S -modules F^k define a filtration, we need only show that $\alpha_k \in F^k$ for all $1 \leq k \leq m$. For this, relation (3.1) gives

$$D^k - \gcd(D^k, D^{k+1}) = D_{k+1}^k - \gcd(D_{k+1}^k, D_k^{k+1}), \quad (3.4)$$

and hence

$$\frac{f^{k,k+1}}{f^{k+1}} = \frac{\text{lcm}(f^k, f^{k+1})}{f^{k+1}} = \frac{f^k}{\gcd(f^k, f^{k+1})} = \frac{f_{k+1}^k}{\gcd(f_{k+1}^k, f_k^{k+1})}.$$

Similarly, we have $f^{k,k+1}/f^k = f_{k+1}^{k+1}/\gcd(f_{k+1}^k, f_k^{k+1})$. Therefore

$$\alpha_k = \gcd(f_{k+1}^k, f_k^{k+1}) \left(\frac{f_{k+1}^{k+1}}{f_{k+1}^k} \mathbf{e}_{k+1} - \frac{f_k^{k+1}}{f_k^k} \mathbf{e}_k \right) = \gcd(f_{k+1}^k, f_k^{k+1}) \beta_k \quad (3.5)$$

for $1 \leq k \leq m$ as required. To prove part (1), we first note that

$$\frac{F^k}{F^{k-1}} \cong \frac{\langle \beta_k \rangle / (\langle \beta_1, \dots, \beta_{k-1}, \alpha_{k+1}, \dots, \alpha_m \rangle \cap \langle \beta_k \rangle)}{\langle \alpha_k \rangle / (\langle \beta_1, \dots, \beta_{k-1}, \alpha_{k+1}, \dots, \alpha_m \rangle \cap \langle \alpha_k \rangle)}.$$

In order to compute this quotient, it suffices, in view of (3.5) and the remarks at the beginning of the proof of Proposition 2.8, to determine a set of generators for the module of syzygies on $\beta_1, \dots, \beta_k, \alpha_{k+1}, \dots, \alpha_m$ for $1 \leq k \leq m$. Proceeding exactly as in the proof of Lemma 2.2, we find that this module is cyclic with generator

$$\sigma_0 := -\frac{\text{lcm}(f^{1,\dots,m}, g^{k+1,k+2}, \dots, g^{m,1})}{f^{1,m}} \varepsilon_m + \sum_{j=1}^{m-1} \frac{\text{lcm}(f^{1,\dots,m}, g^{k+1,k+2}, \dots, g^{m,1})}{f^{j,j+1}} \varepsilon_j, \quad (3.6)$$

where we have set $g^{i,i+1} := \gcd(f_{i+1}^i, f_i^{i+1})$ for $k+1 \leq i \leq m$. Ignoring for now the $\text{Cl}(X)$ -grading, we deduce from this that

$$\frac{\langle \beta_k \rangle}{\langle \beta_1, \dots, \beta_{k-1}, \alpha_{k+1}, \dots, \alpha_m \rangle \cap \langle \beta_k \rangle} \cong \frac{S}{\langle \text{lcm}(f^{1,\dots,m}, g^{k+1,k+2}, \dots, g^{m,1}) / f^{k,k+1} \rangle}.$$

and therefore, by virtue of (3.5),

$$\frac{F^k}{F^{k-1}} \cong \frac{S}{\langle \gcd(f_{k+1}^k, f_k^{k+1}), \text{lcm}(f^{1,\dots,m}, g^{k+1,k+2}, \dots, g^{m,1}) / f^{k,k+1} \rangle}$$

which gives the ideal I_k in part (1). For parts (2) and (3), Proposition 2.8(iii) and (iv) respectively determine the ideals I_k for which F^k / F^{k-1} is isomorphic to S / I_k as ungraded rings.

It remains to establish the isomorphism as $\text{Cl}(X)$ -graded rings. In light of the above and isomorphism (2.10), it suffices to show that the degree of β_k is $L_{\mu_k} \otimes L_{\nu_k} \otimes L^{-1}$ for $1 \leq k \leq n$. For each $1 \leq k \leq n$, multiplication by the monomials f^{μ_k} and f^{ν_k} define $\text{Cl}(X)$ -graded maps $S \rightarrow S(L^{-1} \otimes L_{\mu_k})$ and $S \rightarrow S(L^{-1} \otimes L_{\nu_k})$ respectively. Tensoring each map with $S(L_{\mu_k}^{-1} \otimes L_{\nu_k}^{-1} \otimes L)$ yields $\text{Cl}(X)$ -graded maps $S(L_{\mu_k}^{-1} \otimes L_{\nu_k}^{-1} \otimes L) \rightarrow S(L_{\nu_k}^{-1})$ and $S(L_{\mu_k}^{-1} \otimes L_{\nu_k}^{-1} \otimes L) \rightarrow S(L_{\mu_k}^{-1})$ which, in turn, can be combined to form a $\text{Cl}(X)$ -graded map

$$S(L_{\mu_k}^{-1} \otimes L_{\nu_k}^{-1} \otimes L) \longrightarrow \bigoplus_{j=1}^m S(L_j^{-1}),$$

whose image in $\bigoplus_{j=1}^m S(L_j^{-1})$ is generated by the element $f^{\mu_k} \mathbf{e}_{\nu_k} - f^{\nu_k} \mathbf{e}_{\mu_k}$. Twisting further by $S(\mathcal{O}_X(-\gcd(D^{\mu_k}, D^{\nu_k})))$ determines a $\text{Cl}(X)$ -graded map

$$S(L_{\mu_k}^{-1} \otimes L_{\nu_k}^{-1} \otimes L(-\gcd(D^{\mu_k}, D^{\nu_k}))) \longrightarrow \bigoplus_{j=1}^m S(L_j^{-1})$$

whose image is generated by the element

$$\frac{f^{\mu_k}}{\gcd(f^{\mu_k}, f^{\nu_k})} \mathbf{e}_{\nu_k} - \frac{f^{\nu_k}}{\gcd(f^{\mu_k}, f^{\nu_k})} \mathbf{e}_{\mu_k}. \quad (3.7)$$

To prove the claim it remains to show that (3.7) coincides with β_k , but this is immediate since $f^{\mu_k} / \gcd(f^{\mu_k}, f^{\nu_k}) = \text{lcm}(f^{\mu_k}, f^{\nu_k}) / f^{\nu_k}$ and $f^{\nu_k} / \gcd(f^{\mu_k}, f^{\nu_k}) = \text{lcm}(f^{\mu_k}, f^{\nu_k}) / f^{\mu_k}$. \square

For $1 \leq k \leq n$, each of the generators of I_k listed in Proposition 3.1 is a monomial in the Cox ring S of X , so its divisor of zeros is an effective torus-invariant divisor in X .

Definition 3.2. For each $1 \leq k \leq n$, define a subscheme $Z_k \subset X$ to be the scheme-theoretic intersection of a set of effective divisors depending on k as follows:

- (i) for $1 \leq k \leq m$, the set comprises only two divisors, namely $\gcd(D_{k+1}^k, D_k^{k+1})$ and

$$\text{lcm}(D^1, \dots, D^m, \gcd(D_{k+2}^{k+1}, D_{k+1}^{k+2}), \dots, \gcd(D_1^m, D_m^1)) - \text{lcm}(D^k, D^{k+1});$$

- (ii) for $m+1 \leq k \leq 2m-3$, the divisors are $\text{lcm}(D^1, D^{\nu_k}, D^{\nu_k+1}, \dots, D^m) - \text{lcm}(D^{\nu_k}, D^{\nu_k+1})$ and $\text{lcm}(D^1, D^{\nu_k}, D^{\nu_k+1}) - \text{lcm}(D^{\nu_k}, D^{\nu_k+1})$;
- (iii) for $2m+2 \leq k \leq n$, there are μ_k such divisors, namely $\text{lcm}(D^\mu, D^{\mu_k}, D^{\nu_k}) - \text{lcm}(D^{\mu_k}, D^{\nu_k})$ for $\mu \in \{1, \dots, \mu_k - 1\} \cup \{\nu_k - 1\}$.

These subschemes $Z_1, \dots, Z_n \subset X$ enable us to formulate the main theorem of this paper.

Theorem 3.3. *Let T^\bullet be the total complex arising from the wheel of invertible sheaves in Figure 4. Then:*

- (1) $H^0(T^\bullet) \cong \mathcal{O}_Z \otimes L$ where Z is the scheme-theoretic intersection of D^1, \dots, D^m ;
- (2) $H^{-1}(T^\bullet)$ has an n -step filtration $\text{im}(d^2) = F^0 \subseteq F^1 \subseteq \dots \subseteq F^{n-1} \subseteq F^n = \ker(d^1)$ where, for $1 \leq k \leq n$ and for the permutation $\tau_k = (\mu_k, \nu_k)$, we have

$$F^k / F^{k-1} \cong \mathcal{O}_{Z_k} \otimes L_{\mu_k} \otimes L_{\nu_k} \otimes L^{-1}(\text{gcd}(D^{\mu_k}, D^{\nu_k})); \quad (3.8)$$

- (3) $H^{-2}(T^\bullet) \cong \mathcal{O}_D \otimes L(D)$ where $D = \text{gcd}(D_{1,2}, D_{2,3}, \dots, D_{m,1})$;
- (4) $H^{-3}(T^\bullet) \cong 0$.

Proof. Every quasicoherent sheaf on X corresponds to a $\text{Cl}(X)$ -graded S -module by Cox [5] and Mustařă [9], so part (2) then follows from Proposition 3.1 and Definition 3.2. For part (1), note that $H^0(T^\bullet)$ is the cokernel of $\bigoplus_i \mathcal{O}_X(-D^i) \otimes L \hookrightarrow \mathcal{O}_X \otimes L$, namely the sheaf $\mathcal{O}_Z \otimes L$ where Z is the scheme-theoretic intersection of D^1, \dots, D^m . For part (4), every nonzero map between invertible sheaves is injective, so $H^{-3}(T^\bullet) \cong 0$. It remains to prove part (3). The proof of the analogous statement from [3, Lemma 3.1] does not immediately extend to our setting, as was the case with parts (1) and (4) above, but we can nevertheless adapt the argument as follows.

We claim first that if the greatest common divisor D is zero then $H^{-2}(T^\bullet) \cong 0$. We need only show that complex (3.3) has no cohomology in degree -2 . Indeed, suppose $\eta = \sum_{j=1}^m u_j \mathbf{e}_{j,j+1}$ lies in the kernel of φ^2 , so

$$0 = \varphi^2(\eta) = \sum_{j=1}^m u_j (f_{j+1}^j \mathbf{e}_{j+1} - f_j^{j+1} \mathbf{e}_j).$$

This translates into the following set of equations:

$$u_{j-1} f_j^{j-1} = u_j f_j^{j+1}, \quad 1 \leq j \leq m.$$

By relation (3.2) we have $f_j^{j-1} f_{j-1,j} = f_j^{j+1} f_{j,j+1}$ for $1 \leq j \leq m$. Consequently, we find that

$$u_{j-1} f_{j,j+1} = u_j f_{j-1,j}, \quad 1 \leq j \leq m. \quad (3.9)$$

We claim that $f_{j,j+1}$ divides u_j for all $1 \leq j \leq m$. It suffices to prove that $f_{1,2}$ divides u_1 by virtue of (3.9). Let x_i be a prime factor of $f_{1,2}$ with multiplicity p . Since by assumption $\text{gcd}(f_{1,2}, f_{2,3}, \dots, f_{m,1}) = 1$, it follows that x_i^p does not divide $f_{\nu,\nu+1}$ for some $\nu \neq 1$. Appealing to (3.9) once again, we find that $u_1 f_{\nu,\nu+1} = u_\nu f_{1,2}$, and thus x_i^p divides $u_1 f_{\nu,\nu+1}$. Since S is a UFD, this means that x_i^p divides u_1 , which in turn implies that $f_{1,2}$ divides u_1 . If we now set $u := u_1 / f_{1,2}$, then equations (3.9) give

$$u = \frac{u_1}{f_{1,2}} = \frac{u_2}{f_{2,3}} = \dots = \frac{u_m}{f_{m,1}},$$

from which it follows that $\eta = u \sum_{j=1}^m f_{j,j+1} \mathbf{e}_{j,j+1}$. Thus, η lies in the image of φ^3 , so the complex (3.3) has no cohomology in degree -2 as required.

To complete the proof of part (3), suppose $D \neq 0$. We can factor $d^3: T^{-3} \rightarrow T^{-2}$ as a map $L \rightarrow L(D)$ followed by a map with no common divisors. By the above argument, the image of $L(D)$ under this map equals the kernel of $d^2: T^{-2} \rightarrow T^{-1}$. Therefore $H^{-2}(T^\bullet)$ can be identified with the cokernel of $L \rightarrow L(D)$, which is $\mathcal{O}_D \otimes L(D)$. \square

Remark 3.4. For $m = 3$, the statement of Theorem 3.3 recovers the main technical result¹ from the paper of Cautis–Logvinenko [3, Lemma 3.1]. Parts (1), (3), (4) of Theorem 3.3 evidently generalise the analogues from *loc. cit.*. As for $H^{-1}(T^\bullet)$, Theorem 3.3(2) gives a 3-step filtration $\text{im}(d^2) = F^0 \subseteq F^1 \subseteq F^2 \subseteq F^3 = \ker(d^1)$ and to compare the successive quotients to those of *loc. cit.* we compute F^2/F^1 . Since $\tau_2 = (2, 3)$, Theorem 3.3(2) shows that

$$F^2/F^1 \cong \mathcal{O}_{Z_2} \otimes L_2 \otimes L_3 \otimes L^{-1}(\gcd(D^2, D^3)),$$

where Z_2 is the intersection of $\gcd(D_3^2, D_2^3)$ and $\text{lcm}(D^1, D^2, D^3, \gcd(D_1^3, D_3^1)) - \text{lcm}(D^2, D^3)$. A direct computation shows that the relation defined by the generator σ_0 from (3.6) is

$$\frac{f_1^3}{\gcd(f_1^3, \tilde{f}_1^2)}\beta_1 + \frac{\tilde{f}_2^1 f_1^3}{\gcd(f_1^3, \tilde{f}_1^2) \tilde{f}_2^3}\beta_2 - \frac{\tilde{f}_1^2}{\gcd(f_1^3, \tilde{f}_1^2)}\alpha_3 = 0,$$

where $\tilde{f}_j^i = f_j^i / \gcd(f_j^i, f_i^j)$. Since $k = 2$, the coefficient of β_2 coincides with the generator $\text{lcm}(f^{1,2,3}, \gcd(f_1^3, f_3^1)) / f^{2,3}$ of the ideal I_2 . In particular, the scheme Z_2 is the intersection of $\gcd(D_3^2, D_2^3)$ and $\tilde{D}_2^1 + D_1^3 - \tilde{D}_2^3 - \gcd(D_1^3, \tilde{D}_1^2)$, where \tilde{D}_j^i is the divisor of zeros of the function \tilde{f}_j^i . Permutations are listed as $\tau_1 = (1, 2)$, $\tau_2 = (3, 1)$, $\tau_3 = (2, 3)$ in [3], so after applying permutation $(1, 2, 3)$ to our indices, we need only invoke the identity

$$\tilde{D}_3^2 + D_2^1 - \tilde{D}_3^1 - \gcd(D_2^1, \tilde{D}_2^3) = D^2 + \text{lcm}(D_2^1, \tilde{D}_2^3) - D^3 - \tilde{D}_3^1$$

from [3, p206], we see that Z_2 is the scheme in the second bullet point of [3, Lemma 3.1(2)]. To compare the sheaves, equation (3.4) gives $\gcd(D^2, D^3) = D^2 + \gcd(D_3^2, D_2^3) - D_3^2$, and $\mathcal{O}(D^2) = L_2^{-1} \otimes L$ and $\mathcal{O}(-D_3^2) \cong L_3^{-1} \otimes L_{2,3}$ hence

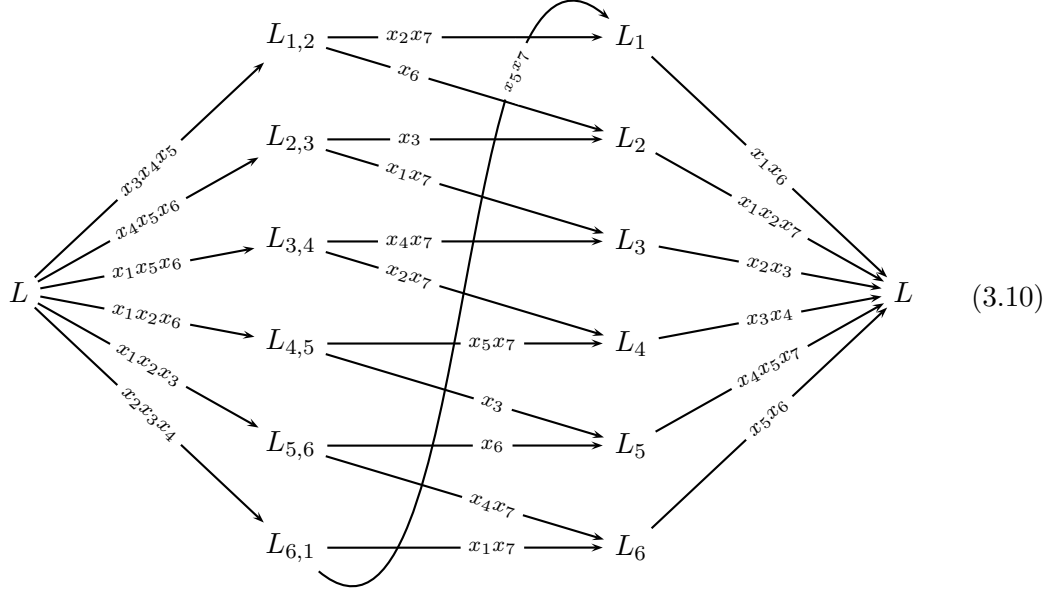
$$\begin{aligned} L_2 \otimes L_3 \otimes L^{-1}(\gcd(D^2, D^3)) &\cong L_2 \otimes L_3 \otimes L^{-1}(\gcd(D_3^2, D_2^3)) \otimes L_2^{-1} \otimes L \otimes L_3^{-1} \otimes L_{2,3} \\ &\cong L_{2,3}(\gcd(D_2^3, D_3^2)). \end{aligned}$$

Again, apply the permutation $(1, 2, 3)$ to the indices to recover the sheaf from the second bullet point of [3, Lemma 3.1(2)].

Example 3.5. Let T^\bullet be the total complex of the diagram (3.10) below. With the notation above, the generators $\beta_1, \dots, \beta_{15}$ of $\ker(d^1)$ are

$$\begin{aligned} \beta_1 &= -x_2 x_7 \mathbf{e}_1 + x_6 \mathbf{e}_2, & \beta_6 &= x_5 \mathbf{e}_1 - x_7 \mathbf{e}_6, & \beta_{11} &= -x_4 x_5 \mathbf{e}_2 + x_1 x_2 \mathbf{e}_5, \\ \beta_2 &= -x_3 \mathbf{e}_2 + x_1 x_7 \mathbf{e}_3, & \beta_7 &= -x_2 x_3 \mathbf{e}_1 + x_1 x_6 \mathbf{e}_3, & \beta_{12} &= -x_5 x_6 \mathbf{e}_2 + x_1 x_2 x_7 \mathbf{e}_6, \\ \beta_3 &= -x_4 \mathbf{e}_3 + x_2 \mathbf{e}_4, & \beta_8 &= -x_3 x_4 \mathbf{e}_1 + x_1 x_6 \mathbf{e}_4, & \beta_{13} &= -x_4 x_5 x_7 \mathbf{e}_3 + x_2 x_3 \mathbf{e}_5, \\ \beta_4 &= -x_5 x_7 \mathbf{e}_4 + x_3 \mathbf{e}_5, & \beta_9 &= -x_4 x_5 x_7 \mathbf{e}_1 + x_1 x_6 \mathbf{e}_5, & \beta_{14} &= -x_5 x_6 \mathbf{e}_3 + x_2 x_3 \mathbf{e}_6, \\ \beta_5 &= -x_6 \mathbf{e}_5 + x_4 x_7 \mathbf{e}_6, & \beta_{10} &= -x_3 x_4 \mathbf{e}_2 + x_1 x_2 x_7 \mathbf{e}_4, & \beta_{15} &= -x_5 x_6 \mathbf{e}_4 + x_3 x_4 \mathbf{e}_6. \end{aligned}$$

¹The proof of Lemma 3.1(2) from *loc. cit.* asserts that certain elements $\beta_1, \beta_2, \beta_3$ generate the kernel of a map. This is true if the functions f^1, f^2, f^3 are monomials, so the result is valid without further restriction if the smooth separated scheme under consideration is a toric variety.



It is easy to see that the relations

$$\begin{aligned}
\beta_9 &= -x_4x_7\beta_6 - x_1\beta_5, \\
\beta_{10} &= x_4\beta_2 + x_1x_7\beta_3, \\
\beta_{12} &= -x_5\beta_1 - x_2x_7\beta_6, \\
\beta_{13} &= x_5x_7\beta_3 + x_2\beta_4,
\end{aligned}$$

hold, so the successive quotients F^k/F^{k-1} vanish for $k = 9, 10, 12, 13$. In addition, the generators $\alpha_1, \dots, \alpha_6$ of $\text{im}(d^2)$ satisfy $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = x_7\beta_3$, $\alpha_4 = \beta_4$, $\alpha_5 = \beta_5$ and $\alpha_6 = x_7\beta_6$, so F^k/F^{k-1} also vanishes for $k = 1, 2, 4, 5$. We now analyse three nonvanishing quotients F^k/F^{k-1} to illustrate part (2) of Theorem 3.3. First consider the case $k = 3$. The corresponding transposition $\tau_3 = (3, 4)$ determines $\gcd(D^3, D^4) = E_3$, so

$$F^3/F^2 \cong \mathcal{O}_{Z_3} \otimes L_3 \otimes L_4 \otimes L^{-1}(E_3)$$

where, according to Definition 3.2(i), Z_3 is the scheme-theoretic intersection of the divisors $\gcd(D_4^3, D_3^4) = E_7$ and

$$\text{lcm}(D^1, D^2, D^3, D^4, D^5, D^6, \gcd(D_5^4, D_4^5), \gcd(D_6^5, D_5^6), \gcd(D_1^6, D_6^1)) - \text{lcm}(D^3, D^4) = E_{1567}.$$

In particular, $\text{supp}(\mathcal{O}_{Z_3}) = E_7$. Now consider the case $k = 7$. The corresponding transposition $\tau_7 = (1, 3)$ determines $\gcd(D^1, D^3) = 0$, so

$$F^7/F^6 \cong \mathcal{O}_{Z_7} \otimes L_1 \otimes L_3 \otimes L^{-1}$$

where, according to Definition 3.2(ii), Z_7 is the scheme-theoretic intersection of the divisors $\text{lcm}(D^1, D^2, D^3) - \text{lcm}(D^1, D^3) = E_7$ and $\text{lcm}(D^1, D^3, D^4, D^5, D^6) - \text{lcm}(D^1, D^3) = E_{457}$, giving $Z_7 = E_7 \cap E_{457}$ and $\text{supp}(\mathcal{O}_{Z_7}) = E_7$. Finally, consider the case $k = 15$ for which the corresponding transposition $\tau_{15} = (4, 6)$ determines $\gcd(D^4, D^6) = 0$, so

$$F^{15}/F^{14} \cong \mathcal{O}_{Z_{15}} \otimes L_4 \otimes L_6 \otimes L^{-1}$$

where, according to Definition 3.2(iii), Z_{15} is the scheme-theoretic intersection of the divisors $\text{lcm}(D^\mu, D^4, D^6) - \text{lcm}(D^4, D^6)$ for $\mu = 1, 2, 3, 5$, giving $Z_{15} = E_1 \cap E_{127} \cap E_2 \cap E_7$. In particular, the support of $\mathcal{O}_{Z_{15}}$ is the torus-invariant point $E_1 \cap E_2 \cap E_7$ in Y .

Notice that $\gcd(D^1, D^2, D^3, D^4, D^5, D^6) = 0$ and $\gcd(D_{1,2}, D_{2,3}, D_{3,4}, D_{4,5}, D_{5,6}, D_{6,1}) = 0$. Parts (1) and (3) of Theorem 3.3 imply that $H^0(T^\bullet) \cong 0$ and $H^{-2}(T^\bullet) \cong 0$, so the complex T^\bullet has cohomology concentrated in degree -1 .

REFERENCES

- [1] Dave Bayer, Irena Peeva, and Bernd Sturmfels. Monomial resolutions. *Math. Res. Lett.*, 5(1-2):31–46, 1998.
- [2] Raf Bocklandt, Alastair Craw, and Alexander Quintero Vélez, 2012. Work in preparation.
- [3] Sabin Cautis and Timothy Logvinenko. A derived approach to geometric McKay correspondence in dimension three. *J. Reine Angew. Math.*, 636:193–236, 2009.
- [4] Sabin Cautis, Alastair Craw, and Timothy Logvinenko, 2012. *Derived Reid’s recipe for abelian subgroups of $\text{SL}(3, \mathbb{C})$* . Preprint [arXiv:1205.3110](https://arxiv.org/abs/1205.3110).
- [5] David Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4 (1):17–50, 1995.
- [6] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
- [7] Daniel Grayson and Michael Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at www.math.uiuc.edu/Macaulay2/.
- [8] Martin Kreuzer and Lorenzo Robbiano. *Computational commutative algebra. 1*. Springer-Verlag, Berlin, 2000.
- [9] Mircea Mustață. Vanishing theorems on toric varieties. *Tohoku Math. J. (2)*, 54 (3):451–470, 2002.
- [10] Hidefumi Ohsugi and Takayuki Hibi. Toric ideals generated by quadratic binomials. *J. Algebra*, 218(2):509–527, 1999.
- [11] Rafael Villarreal. *Monomial algebras*, volume 238 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 2001.

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